File

Name: .... Maths Teacher: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 1

TRIAL HSC

2016

Time allowed: 2 hours plus 5 min reading time

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A BOSTES reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1 Multiple Choice
Questions 1-10
10 Marks (allow 15 minutes)

Section II Questions 11-14
60 Marks (allow 1 hour 45 min)

Total Marks 70

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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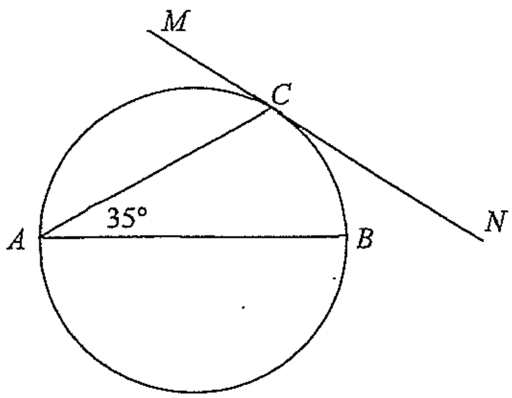
1 In the diagram, AB is a diameter of the circle and MCN is the tangent to the circle at C. $\angle CAB = 35°$. What is the size of $\angle MCA$? **1**



(A) 35°

(B) 45°

(C) 55°

(D) 65°

2 When the polynomial $P(x) = x^3 - 5x^2 + kx + 2$ is divided by $(x+1)$ the remainder is 3. What is the value of k ? **1**

(A) −7

(B) −5

(C) 5

(D) 7

Marks

3 Which of the following is a simplification of $4\log_e \sqrt{e^x}$? **1**

(A) $4\sqrt{x}$

(B) $\frac{1}{2}x$

(C) $2x$

(D) x^2

4 The acute angle between the lines $2x - y = 0$ and $kx - y = 0$ is equal to $\frac{\pi}{4}$. **1**
What is the value of k?

(A) $k = -3$ *or* $k = -\frac{1}{3}$

(B) $k = -3$ *or* $k = \frac{1}{3}$

(C) $k = 3$ *or* $k = -\frac{1}{3}$

(D) $k = 3$ *or* $k = \frac{1}{3}$

5 Which of the following is a simplification of $\dfrac{1-\cos 2x}{\sin 2x}$? **1**

(A) $1 - \cot 2x$

(B) 1

(C) $\cot x$

(D) $\tan x$

6 The statement $7^n - 3^n$ *is always divisible by* 10 is true for **1**

(A) all integers $n \geq 1$

(B) all integers $n \geq 2$

(C) all odd integers $n \geq 1$

(D) all even integers $n \geq 2$

Marks

7 What is the value of $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$? **1**

(A) $\frac{\pi}{6}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{2}$

8 The radius r of a circle is increasing at a constant rate of $0\cdot1\text{ cm s}^{-1}$. **1**
What is the rate at which the area of the circle is increasing when $r = 10$ cm ?

(A) $\pi\text{ cm}^2\text{ s}^{-1}$
(B) $2\pi\text{ cm}^2\text{ s}^{-1}$
(C) $10\pi\text{ cm}^2\text{ s}^{-1}$
(D) $20\pi\text{ cm}^2\text{ s}^{-1}$

9 If $x+\frac{1}{x}=2$ what is the value of $x^2+\frac{1}{x^2}$? **1**

(A) 2
(B) 4
(C) 6
(D) 8

10 A particle is performing Simple Harmonic Motion in a straight line. In 1 minute of its motion it completes exactly 15 oscillations and travels exactly 120 metres. What is the amplitude of the motion? **1**

(A) 2 metres
(B) 4 metres
(C) 8 metres
(D) 16 metres

Section II

60 Marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer the questions in writing booklet provided. Use a new page for each question.
In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Marks

a) Solve $\frac{2x+1}{x-2} \geq 1$ 2

b) P divides AB externally in the ratio 3:2.
Find the co-ordinates of B given that

A is (-2, 5) and P is (1, 3) 2

c) Solve $cos^2 x + sin x - 1 = 0$

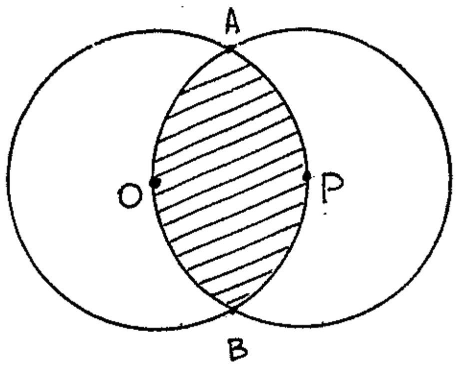
for $0 \leq x \leq 2\pi$ 2

d) Given that $\frac{dy}{dx} = \frac{1}{1+x^2}$ and $x = 1$ when $y = 0$, find y when $x = \sqrt{3}$ 3

e) Differentiate $y = \ln(\sin^{-1} x)$ with respect to x 2

Marks

f) In the diagram below, the two circles are of radius 1 unit and pass through the centres O and P. The circles intersect at A and B.



i) Find the size of angle AOB 1

ii) Find the shaded area in exact form. 3

Question 12 (15 marks) Marks

(Start a new page)

a) Evaluate $\int_0^{\pi/8} cos^2 2x \, dx$ in exact form. 3

b) Evaluate $\int_0^{\pi/4} \sin x . cos^3 x \, dx$ by 3

using the substitution $u = \cos x$, or otherwise.

c) i) Sketch $y = \sin^{-1}(1 - x)$

and state the domain 3

ii) Show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$ 3

iii) Hence, or otherwise, solve the equation

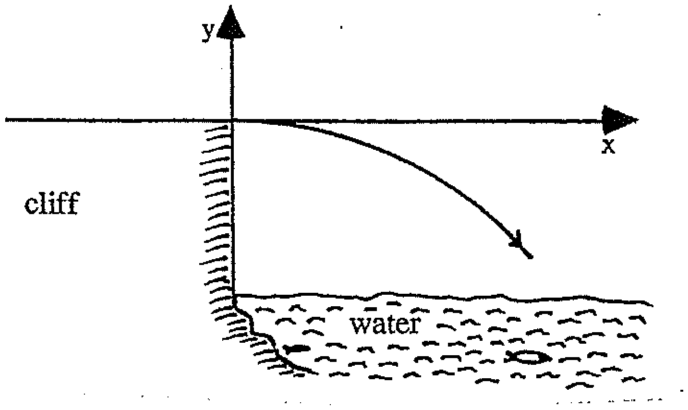
$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1 - x)$ 3

Question 13 (15 marks) Marks

(Start a new page)

a) Use the principle of Mathematical Induction to prove that $7^n + 2$ is divisible by 3 for all positive integers n. 3

b) An object is projected horizontally from the top edge of a vertical cliff 40metres above sea level with a velocity of 40m/s. (Take g = 10 m/s²)



i) Using the top edge of the cliff as origin, show that the parametric equations of the path of the object are:

$$x = 40t \qquad y = -5t^2$$ 2

ii) Calculate when and where the object hits the water. 2

iii) Find the velocity of the object the instant it hits the water. 1

c) The inside of a vessel used for water has the shape of a solid of revolution obtained by the rotation of the parabola $9y = 8x^2$ about the $y-$ axis. The depth of the vessel is 8 cm

i) Prove that a volume of water h cm from its bottom is $\frac{9}{16}\pi h^2$. 1

ii) If water is poured into the vessel at a rate of 20 cm³/sec, find the rate at which the level of water is rising when the vessel is <u>half full</u>. 2

d) The acceleration of a particle is given by $\frac{d^2x}{dt^2} = 16(1+x)$, where x cm is the displacement from the origin. When $t = 0, x = 0$ and $v = 4$ cm/sec.

i) Derive an expression for its velocity in terms of its displacement. 2

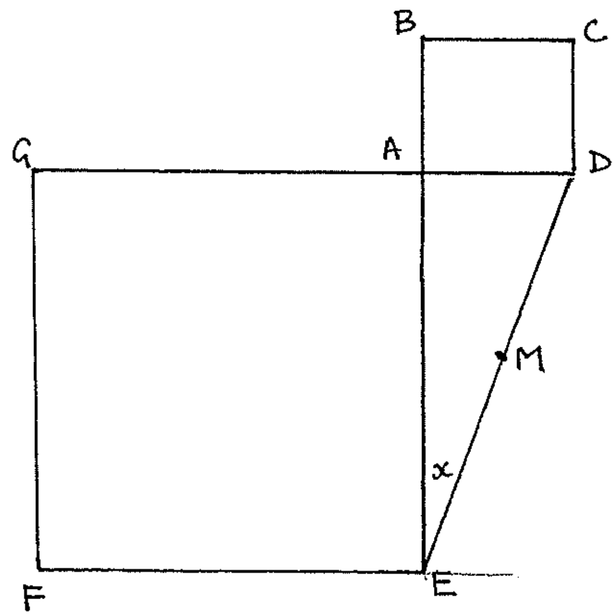
ii) Deduce that its displacement function is $x = e^{4t} - 1$. 2

<u>Question 14</u> (15 Marks) (Start a new page) <u>Marks</u>

a) A particle moves with simple harmonic motion. At at the extremities of the motion the absolute value of the acceleration is 1 cm s^{-2} and when the particle is 3 cm from the centre of motion, the speed is $2\sqrt{2}$ cm s^{-1}. Find the period and amplitude for this motion. 3

b) ABCD and AEFG are two squares of different areas, and $GD \perp BE$. M is the mid point of DE.

Let $\hat{AED} = x$



i) Copy the diagram into your answer book

ii) Give a reason why DE is the diameter of the circle with points A, D and E on its circumference. 1

iii) Prove that BDEG is a cyclic quadrilateral (reasons required) 1

iv) Produce MA to meet BG at T. Prove $MA \perp BG$ (reasons required) 3

c) Two parametric points $P(2p,p^2)$ and $Q(2q,q^2)$ lie on the parabola $x^2 = 4y$, and the line through PQ is parallel to the line $y = mx$.

i) Show that $p + q = 2m$. 1

ii) Derive the equation of the normal to the parabola at the point P. 1

iii) Find the co-ordinates of N, the point of intersection of the normals from P and Q. 2

iv) Determine the locus of N as the line PQ moves parallel to the line $y = mx$. Without further calculations, write any restrictions placed upon the locus of N. 3

S.T.H.S TRIAL H.S.C. Solutions Ext 1 2016.

Section 1

Question	Answer	Solution
	C	$\angle ACB = 90°$ ($\angle$ *in a semi-circle is a right angle*) $\therefore \angle CBA = 55°$ ($\angle$ *sum of* ΔABC *is* 180°) $\therefore \angle MCA = 55°$ (*using alternate segment theorem*)
	A	$P(-1)=3 \Rightarrow -1-5-k+2=3 \quad \therefore k=-7$
	C	$4\log_e\sqrt{e^x} = 4\log_e e^{\frac{1}{2}x} = 4\times\frac{1}{2}x = 2x$
	B	$\tan\frac{\pi}{4} = 1 = \left\lvert\frac{k-2}{1+2k}\right\rvert \quad \therefore \quad 1+2k=k-2,\ k=-3 \quad or \quad 1+2k=-(k-2),\ 3k=1 \quad \therefore\ k=-3 \ or\ k=\frac{1}{3}$
	D	$\frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$
	D	$7^{n+2}-3^{n+2} = 9(7^n-3^n)+40\times 7^n$ and $7^1-3^1=4$, $7^2-3^2=40$. Since the prime factors of 10 are not factors of 9, and 40 is divisible by 10, by the process of Mathematical Induction, the statement cannot be true for odd positive integers n, but is true for even positive integers n.
	C	$\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^2 = \sin^{-1}1 - \sin^{-1}\frac{1}{2} = \frac{\pi}{2}-\frac{\pi}{6} = \frac{\pi}{3}$
	B	$A=\pi r^2 \quad \therefore \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\times 10\times 0{\cdot}1 = 2\pi \quad$ *Ans.* 2π cm^2 s^{-1}
	A	$x^2+\frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2.x.\frac{1}{x} = 2^2-2 = 2$
10	A	If the amplitude is A metres, then $15\times 4A = 120 \quad \therefore A=2 \quad$ *Ans.* 2 metres

Section II

Question 11

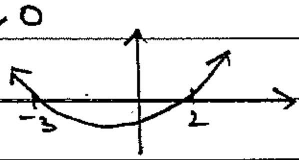
a) $\frac{2x+1}{x-2} \geq 1$

$(x-2)(2x+1) \geq (x-2)^2$

$(x-2)(2x+1) - (x-2)^2 \geq 0$

$(x-2)(2x+1-(x-2)) \geq 0$

$(x-2)(x+3) \geq 0$



$\therefore x > 2,\ x \leq -3$

b) $A(-2,5) \quad B(x,y)$

$-3:2 \quad P(1,3)$

$\frac{(2\times -2)+-3x}{-1} = 1 \quad \therefore -4-3x=-1$

$x=-1$

$\frac{(2\times 5)+-3y}{-1} = 3 \quad \therefore 10-3y=-3$

$y = 13/3$

$\therefore B(-1, 13/3)$

c) $\cos^2 x + \sin x - 1 = 0$

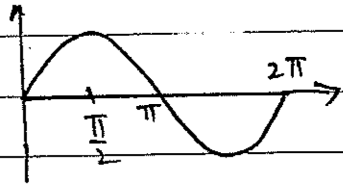
$\cos^2 x + \sin x - 1 = 0$

$1-\sin^2 x + \sin x - 1 = 0$

$\sin x - \sin^2 x = 0$

$\sin x(1-\sin x) = 0$

$\sin x = 0 \qquad \sin x = 1$



$\therefore x = 0, \pi, 2\pi$ and $\frac{\pi}{2}$

d) $\frac{dy}{dx} = \frac{1}{1+x^2}$

$\therefore y = \tan^{-1}x + c$

sub $(1,0)$

$0 = \tan^{-1}1 + c$

$\therefore c = -\pi/4$

$y = \tan^{-1}x - \pi/4$

sub $x=\sqrt{3}$

$y = \tan^{-1}\sqrt{3} - \frac{\pi}{4}$

$y = \pi/3 - \pi/4$

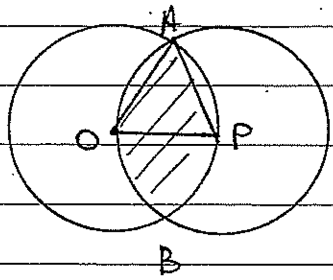
$\therefore y = \frac{\pi}{12}$

e) Let $u = \sin^{-1}x \quad \therefore y = \ln u$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{dy}{du} = \frac{1}{u}$

$\therefore \frac{dy}{dx} = \frac{1}{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$

f)



i) $\triangle OAP$ is equilateral

$\therefore \hat{AOP} = \pi/3$

$\therefore \hat{AOB} = \frac{2\pi}{3}$

ii)

Shaded area $= 2 \times \frac{1}{2} \times 1^2 \times \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= \frac{2\pi}{3} - \sin\frac{\pi}{3}$

$= \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ units^2

Question 12

a) $\int_0^{\pi/8} \cos^2 2x \, dx$

since $\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$= \frac{1}{2}\int_0^{\pi/8} (\cos 4x + 1)\, dx$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]_0^{\pi/8}$

$= \frac{1}{2}\left[\frac{1}{4}\cancel{\sin\frac{\pi}{2}}^{1} + \frac{\pi}{8}\right]$

$= \frac{1}{8} + \frac{\pi}{16}$

b) $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\therefore dx = \frac{du}{-\sin x}$

$x = \pi/4 \quad u = \cos\pi/4$

$\therefore u = 1/\sqrt{2}$

$x = 0 \quad u = \cos 0$

$\therefore u = 1$

$\therefore \int_0^{\pi/4} \sin x . \cos^3 x \, dx$

$\int_1^{1/\sqrt{2}} \sin x . u^3 . \frac{du}{-\sin x}$

$- \int_{1/\sqrt{2}}^{1} \cancel{\sin x} . u^3 \frac{du}{-\cancel{\sin x}}$

$\int_{1/\sqrt{2}}^{1} u^3 \, du$

$= \left[\frac{u^4}{4}\right]_{1/\sqrt{2}}^{1}$

$= \frac{1}{4} - \frac{1}{16}$

$= \frac{3}{16}$

QUESTION 13

a) Step ① Show true for $n=1$

$7^1+2=9$ div by 3

Step ② Assume true for $n=k$ some +ve integer

$7^k+2=3M$ (where M is an intege

Step ③ Show true for $n=k+1$

$$7^{k+1}+2 = 7^k.7+2$$
$$= (3M-2)7+2 \quad \text{from step ②}$$
$$= 21M-14+2$$
$$= 21M-12$$
$$= 3(7M-4)$$

Step ④ Since true for $n=1$ and if assumed true for $n=k$ (some +ve integer) we have shown true for $n=k+1$ $\therefore$ true for all +ve integers $(n \geq 1)$

b) i)

$\ddot{x}=0$ | $\ddot{y}=-10$

$\dot{x}=c_1$ | $\dot{y}=-10t+k_1$

$\therefore \dot{x}=40$ $\quad c_1=40$ | $\therefore \dot{y}=-10t$ $\quad k_1=0$

$x=40t+c_2$ | $y=-5t^2+k_2$

$\therefore x=40t$ $\quad c_2=0$ | $\therefore y=-5t^2$ $\quad k_2=0$

using initially $t=0$, $x=0$ & $y=0$

$\dot{y}=0 \quad \dot{x}=40$

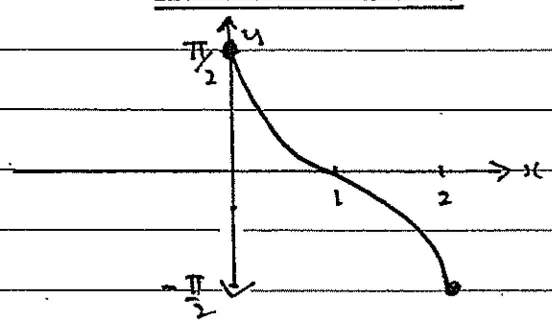
ii) hits water if $y=-40$

$$\therefore -40=-5t^2$$
$$t^2=8 \qquad t \geq 0$$
$$\therefore t=\sqrt{8}$$
$$t=2\sqrt{2}\text{ sec}$$
$$x=2\sqrt{2}\times 40 = 80\sqrt{2}\text{ m from base}$$

c) $y=\sin^{-1}(1-x)$

i)
$$-1 \leq 1-x \leq 1$$
$$-2 \leq -x \leq 0$$

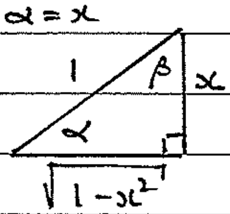
Domain $\therefore 0 \leq x \leq 2$



ii) Let

$\alpha=\sin^{-1}x \qquad \beta=\cos^{-1}x$

$\therefore \sin\alpha=x \qquad \cos\beta=x$



since $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

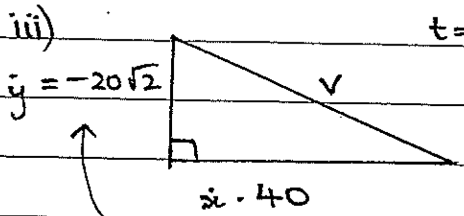
$$= x.x - \frac{\sqrt{1-x^2}}{1}.\frac{\sqrt{1-x^2}}{1}$$
$$= x^2-(1-x^2)$$
$$\sin(\alpha-\beta) = 2x^2-1$$
$$\therefore \alpha-\beta = \sin^{-1}(2x^2-1)$$
$$\therefore \sin(\sin^{-1}x-\cos^{-1}x) = 2x^2-1$$

iii) $\sin^{-1}x-\cos^{-1}x = \sin^{-1}(1-x)$

$$\sin(\sin^{-1}x-\cos^{-1}x) = 1-x$$
$$2x^2-1 = 1-x$$
$$2x^2+x-2=0$$
$$x=\frac{-1\pm\sqrt{17}}{4} \qquad \text{since } 0 \leq x \leq 2$$

only solution $x=\dfrac{-1+\sqrt{17}}{4} \doteqdot 0.78$

iii) $t = 2\sqrt{2}$



since $\dot{y} = -10t$

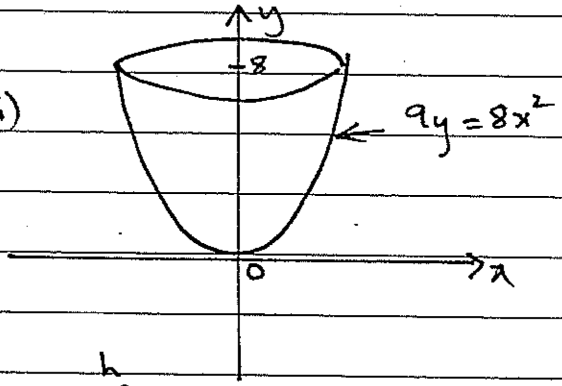
$v^2 = (20\sqrt{2})^2 + (40)^2$

$= 2400$

$\underline{\underline{v = 20\sqrt{6}\ \text{m/s}}}$

c)

i)



$V = \pi \int_0^h \frac{9}{8} y \, dy$

$= \pi \left[\frac{9y^2}{16} \right]_0^h$

$\underline{\underline{V = \frac{9\pi h^2}{16} \text{ unit}^3}}$

ii) $\frac{dV}{dt} = 20\text{cm}^3/\text{sec}$, $\frac{dV}{dh} = \frac{9\pi h}{8}$

Full if $h = 8$ $V = 36\pi$

$\therefore$ $\frac{1}{2}$ full $V = 18\pi$

$18\pi = \frac{9\pi h^2}{16}$

$32 = h^2$

$\underline{\underline{\therefore h = 4\sqrt{2}}}$

$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}$

$= 20 \times \frac{8}{9\pi h}$

$= \frac{160}{9\pi h}$ sub $h = 4\sqrt{2}$

$\underline{\underline{\frac{dh}{dt} = \frac{40}{9\pi\sqrt{2}} \text{ cm/s} \quad \text{OR} \quad \frac{20\sqrt{2}}{9\pi}}}$

d) $\ddot{x} = 16(1+x)$ $t=0$

$x=0$ $v=4$

i)

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 16 + 16x$

$\frac{1}{2}v^2 = 16x + 8x^2 + c$

$v^2 = 32x + 16x^2 + K$

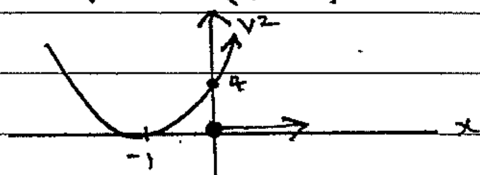
$x=0$ $v=4$

$\therefore 16 = K$

$v^2 = 16x^2 + 32x + 16$

$v^2 = 16(x^2 + 2x + 1)$

$v^2 = 16(x+1)^2$



when $x = 0$ $v = {}^{+}4$ cm/sec

$\underline{\underline{\therefore v = {}^{+}4(x+1)}}$

ii) $\frac{dx}{dt} = 4x + 4$

$\frac{dt}{dx} = \frac{1}{4x+4}$

$t=0$

$\therefore t = \frac{1}{4}\ln(4x+4) + c$ $x=0$

$\therefore c = -\frac{1}{4}\ln(4)$

$t = \frac{1}{4}\ln(4x+4) - \frac{1}{4}\ln 4$

$t = \frac{1}{4}\ln(x+1)$

$$t = \frac{1}{4}\ln(x+1)$$

$$4t = \log_e(x+1)$$

$$e^{4t} = x+1$$

$$\therefore \underline{x = e^{4t} - 1}$$

Question 14

a) $\ddot{x} = -n^2 x$

at extremities $x = a$ $\ddot{x} = -1$, $x = -a$ $\ddot{x} = 1$

$$\therefore 1 = n^2 a$$

$$n^2 = \frac{1}{a} \quad \text{——} \ ①$$

$v^2 = n^2(a^2 - x^2)$ $\quad v = 2\sqrt{2}$ $\quad x = 3$

$$(2\sqrt{2})^2 = n^2(a^2 - 9) \quad \text{——} \ ②$$

sub ① into ②

$$8 = \frac{1}{a}(a^2 - 9)$$

$$8a = a^2 - 9$$

$$a^2 - 8a - 9 = 0$$

$$(a-9)(a+1) = 0$$

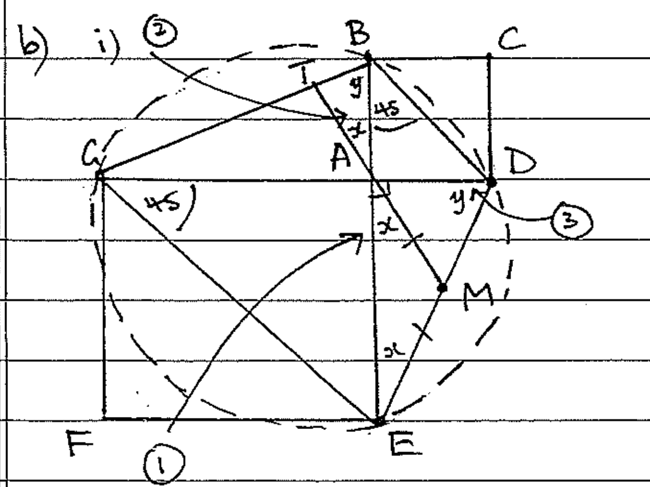
$\therefore$ $a = 9$ only since $a > 0$

$$n = \frac{1}{3}$$

$\therefore$ amplitude is 9 cm

period $\frac{2\pi}{1/3} = 6\pi$ sec

b) i)



ii) DE is the diameter

M is the centre of circle

(angle in semicircle is 90°)

$D\hat{A}E = 90°$ (angle sum of straight line GAD and $G\hat{A}E = 90°$)

iii) $D\hat{B}A = E\hat{G}A = 45°$ (diagonals of square bisect angles)

$D\hat{B}A = E\hat{G}A$ (angles in same segment equal)

$\therefore$ BDEG is a cyclic quad

iv)

- $M\hat{A}E = x$ (opposite equal sides in isosceles $\triangle AME$

(since AM = ME angle in semi circle part ii)

- $T\hat{A}B = x$ (vertically opposite)
- Let $G\hat{D}E = y$

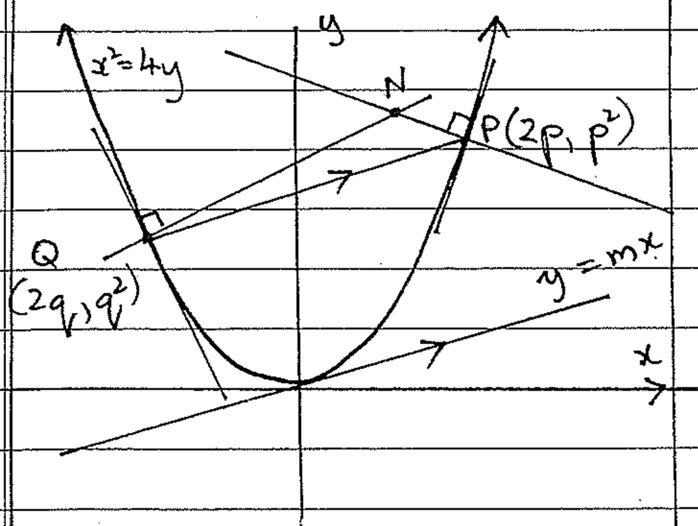
$\therefore G\hat{B}E = y$ (angles in alternate segment) (~~cyclic quad~~ BDEG part iii

since $x + y = 90°$ angle sum $\triangle ADE$

$\therefore B\hat{T}A = 90°$ angle sum $\triangle TBA$

$\therefore$ MA $\perp$ BG

c)



i) $m_{PQ} = \dfrac{p^2 - q^2}{2p-2q}$

$= \dfrac{\cancel{(p-q)}(p+q)}{2\cancel{(p-q)}} = \dfrac{p+q}{2}$

gradient of line $y=mx$ is m

$\therefore \quad m = \dfrac{p+q}{2}$

$\therefore \underline{\underline{2m = p+q}}$

ii) $y = \dfrac{x^2}{4}$

$\dfrac{dy}{dx} = \dfrac{2x}{4} = \dfrac{x}{2} \qquad P(2p, p^2)$

at P $\quad m_T = p \quad \therefore \quad m_N = \dfrac{-1}{p}$

eqn of normal at P

$\underline{\underline{y - p^2 = -\frac{1}{p}(x - 2p)}}$

iii) $y = p^2 - \frac{1}{p}(x-2p)$ normal at P

$y = q^2 - \frac{1}{q}(x-2q)$ normal at Q

sim. eq.

$p^2 - \frac{1}{p}(x-2p) = q^2 - \frac{1}{q}(x-2q)$

$p^2 - \frac{x}{p} + 2 = q^2 - \frac{x}{q} + 2$

$\frac{x}{q} - \frac{x}{p} = q^2 - p^2$

$x\left(\frac{1}{q} - \frac{1}{p}\right) = (q-p)(q+p)$

$x\left(\dfrac{p-q}{pq}\right) = (q-p)(q+p)$

$x = \dfrac{pq(q-p)(q+p)}{(p-q)}$

$\underline{\underline{x = -pq(q+p)}}$

$\therefore y = p^2 - \frac{1}{p}\left(-pq(p+q) - 2p\right)$

$y = p^2 - \frac{1}{p}\left(-p^2q - pq^2 - 2p\right)$

$y = p^2 + pq + q^2 + 2$

$\therefore \underline{\underline{N\left(-pq(q+p),\ p^2+pq+q^2+2\right)}}$

iv) Locus of N

$x = -pq(q+p)$

$y = p^2 + pq + q^2 + 2$

and $p+q = 2m$ *

$\therefore \quad x = -2m\,pq \qquad$ ** $pq = \dfrac{-x}{2m}$

since $(p+q)^2 = p^2 + q^2 + 2pq$

substitute * and **

$(2m)^2 = (p^2+q^2+pq+2) + pq - 2$

$(2m)^2 = y - \dfrac{x}{2m} - 2$

$4m^2 = y - \dfrac{x}{2m} - 2$

$\therefore \underline{\underline{y = \dfrac{x}{2m} + 4m^2 + 2}}$

* N must lie inside the parabola as N is pt on intersecting normals.